

# Lecture 17 - The Secrets we have Swept Under the Rug

Today's lectures examines some of the quirky features of electrostatics that we have neglected up until this point

## A Puzzle...

Let's go back to the basics and consider a point charge  $q$  at the origin. We can apply Gauss's Law over a sphere of radius  $r$  centered at  $q$  to confirm that the radial electric field  $\vec{E} = E[r] \hat{r}$  satisfies  $\frac{q}{\epsilon_0} = \int \vec{E} \cdot d\vec{a} = 4\pi r^2 E[r]$  and hence  $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$ , as expected. However, using the **divergence theorem**, we could rewrite this integral over the surface of our sphere as the volume integral over the interior of the sphere (essentially switching between the integral and differential form of Gauss's Law),

$$\frac{q}{\epsilon_0} = \int (\vec{\nabla} \cdot \vec{E}) dv \quad (1)$$

The divergence of the vector  $\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$  in spherical coordinates is given by

$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial [r^2 A_r]}{\partial r} + \frac{1}{r \sin[\theta]} \frac{\partial [A_\theta \sin[\theta]]}{\partial \theta} + \frac{1}{r \sin[\theta]} \frac{\partial A_\phi}{\partial \phi}$ . But when we evaluate the divergence for the point charge's

electric field  $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$ , we get  $\vec{\nabla} \cdot \vec{E} = 0$ , implying that the right-hand side of Equation (1) equals zero instead of  $\frac{q}{\epsilon_0}$ . Explain what is going on?

### Solution

This problem highlights the subtlety inherent in dealing with point charges, which are inherently strange objects. For example, the electric field infinitely close to a point charge diverges, and the energy required to make a point charge (which can be determined by integrating  $\frac{\epsilon_0}{2} \int E^2 dv$  over all space) similarly blows up. In this problem, we note that  $\vec{\nabla} \cdot \vec{E} = 0$  everywhere in space

...or rather, almost everywhere in space. The differential form of Gauss's Law states that  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ , and because the only charge is a point charge at the origin, it makes sense that  $\vec{\nabla} \cdot \vec{E} = 0$  at all other points in space. Indeed, if we consider any small closed Gaussian surface at any point in space that does not contain the origin, we expect that all electric field lines entering it will also leave it, implying that there is no net flux through the surface and that  $\vec{\nabla} \cdot \vec{E} = 0$ .

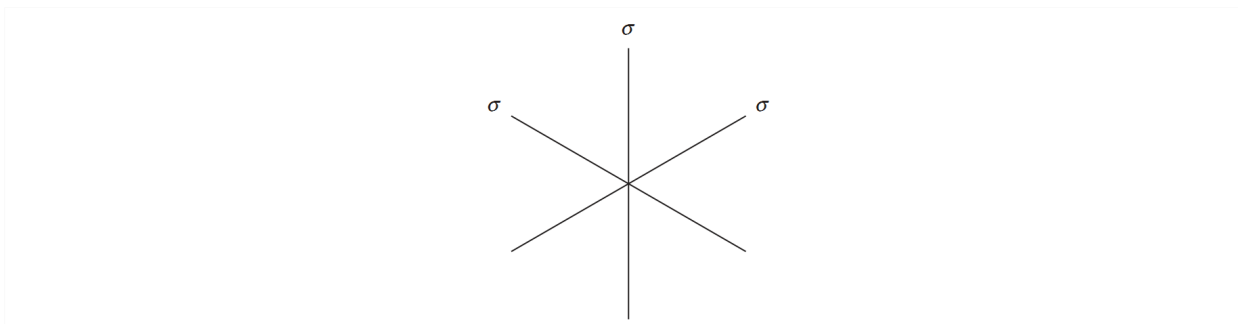
But at the origin, something special happens. At this point, we know that  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ , but what is  $\rho$  (the charge per unit volume)? The charge density of a point charge is infinite, since any box that surrounds the point charge, regardless of how small it is, will contain net charge  $q$ , so  $\rho$  diverges at the origin. It turns out that  $\rho$  blows up as a delta function ( $4\pi \delta^3[\vec{r}]$ ) which you will learn about in junior-level electrodynamics in a way that precisely implies that when you integrate  $\int (\vec{\nabla} \cdot \vec{E}) dv$  over any volume that contains the origin, no matter how small, the right-hand side of Equation (1) will correctly evaluate to  $\frac{q}{\epsilon_0}$ .  $\square$

## Some Interesting Mathematics

### Non-Symmetric Gauss's Law

#### Example

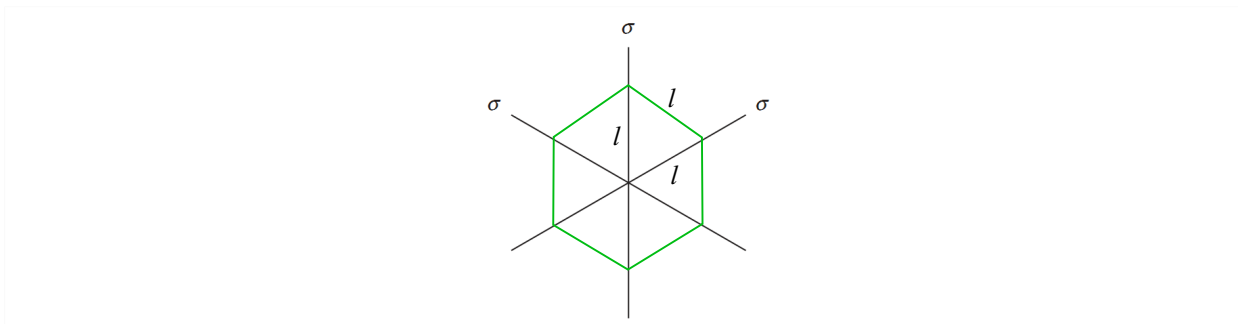
Consider the cross section of three infinite sheets intersecting at equal angles. The sheets all have surface charge density  $\sigma$ .



1. By adding up the fields from the sheets, find the electric field at all points in space.
2. Find the field instead by using Gauss's law. You should explain clearly why Gauss's law is in fact useful in this setup.
3. What is the field in the analogous setup where there are  $N$  sheets instead of three? What is your answer in the  $N \rightarrow \infty$  limit?

#### Solution

1. The electric field from a given sheet points away from the sheet and has uniform magnitude  $\frac{\sigma}{2\epsilon_0}$ . Therefore, in any of the 6 regions of space divided by the sheets, the electric field at every point in this region will be the same. Considering a point on the  $x$ -axis (i.e. directly to the right of the intersection point in the diagram above), the electric field from the nearest two sheets will be  $2 \frac{\sigma}{2\epsilon_0} \sin\left[\frac{\pi}{3}\right] = \frac{\sigma}{2\epsilon_0}$  in the  $x$ -direction. The contribution of the third sheet will also be  $\frac{\sigma}{2\epsilon_0}$  in the  $x$ -direction, leading to an electric field in this entire region of  $\frac{\sigma}{\epsilon_0}$  in the  $x$ -direction. By radial symmetry, this is the magnitude of the electric field in all of the regions, with the direction of it appropriately rotated in each region.
2. We use for our Gaussian surface a regular hexagon centered on the intersection point of the sheet.



Assume that the length of the Gaussian surface between two sheets is  $l$  and that its depth (going into and out of the page) is  $d$ . Because the electric field points perpendicularly outwards against our Gaussian surface, Gauss's Law yields

$$E(6ld) = \frac{(6ld)\sigma}{\epsilon_0} \quad (2)$$

or equivalently  $E = \frac{\sigma}{\epsilon_0}$ , as found above. Note that while Gauss's Law is always valid, it is only useful when you can find a simple surface that is everywhere perpendicular to the electric field and where the electric field has a uniform value.

3. For general  $N$ , we use Gauss's Law (or, for the mathematically minded, set up a summation over pairs of electric sheets as we did in the first part of this problem) using a regular  $2N$ -gon with side length  $l$  and depth  $d$ . The surface area of this  $2N$ -gon is  $(2N) \left(2 \sin\left[\frac{\pi}{2N}\right]\right) ld$  and the charge enclosed is  $(2N) ld \sigma$ . So Gauss's Law yields

$$E(2Nld) = \frac{\sigma(2N)d}{\epsilon_0} \frac{l}{2 \sin\left[\frac{\pi}{2N}\right]} \quad (3)$$

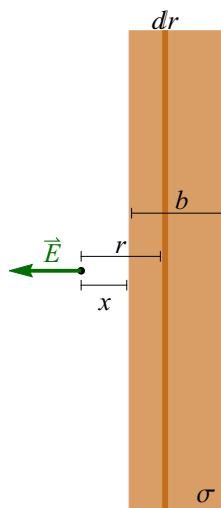
or equivalently  $E = \frac{\sigma}{2 \epsilon_0 \sin\left[\frac{\pi}{2N}\right]}$ . As expected, this is independent of  $l$ . And it agrees with the above result when

$N = 3$ . For large  $N$ ,  $\sin\left[\frac{\pi}{2N}\right] \approx \frac{\pi}{2N}$  so  $E \approx \frac{N}{\pi} \frac{\sigma}{\epsilon_0}$ . In the case of large  $N$ , the sheets are very close to each other, so we effectively have a continuous volume charge distribution that depends on  $r$ . The separation between adjacent sheets grows linearly with  $r$ , so we have  $\rho[r] \propto \frac{1}{r}$ . More precisely, you can easily show that  $\rho[r] = \frac{N\sigma}{\pi r}$ . As a simple exercise, you can use Gauss's Law to easily show that for a volume charge distribution with cylindrical symmetry and a constant electric field that is independent of  $r$ , we must have  $\rho[r] \propto \frac{1}{r}$ .  $\square$

## Electric Field at Corners

### Example

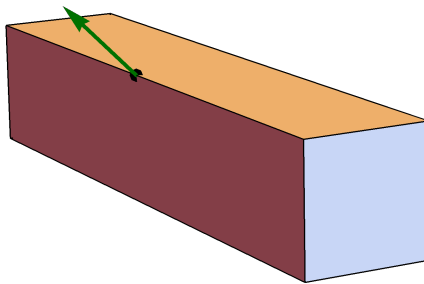
1. Consider an thin sheet of uniform charge density  $\sigma$  (shown below) that extends infinitely in one direction and has a width  $b$  the other direction. What is the electric field at a distance  $x$  from the sheet? What happens as  $x \rightarrow 0$ ?
2. A very long tube has a square cross section and uniform charge density  $\sigma$ . What will be the electric field at a corner? What is the electric field if the tube is a conductor so that the charge is free to flow?



### Solution

Let's start with a brief survey of the second part of the question, that is, of the electric field at the corner of a conducting tube. By symmetry, we know that the electric field must point diagonally away from the central axis of the tube. But this is problematic, since the electric field must point away from the surface of a conductor, and at the corner it seems that the electric field must point in two directions at once (to be normal to both faces it is

touching). Your physics senses should be tingling.



1. To get a handle on the problem, let's first consider the electric field from just one face of the conducting tube, or equivalently from an infinite conducting strip. Let's assume for now that the strip has a uniform surface charge density  $\sigma$ , total width  $b$  and negligible thickness as shown above.

Treating each of the narrow strips as thin rods whose electric field (you will recall from Gauss's Law) equals  $\vec{E}_{\text{rod}} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$ , the total electric field at a distance  $x$  from the side of the strip is given by

$$\vec{E} = \int_x^{x+b} \frac{\sigma dr}{2\pi\epsilon_0 r} = \frac{\sigma}{2\pi\epsilon_0} \text{Log}\left[\frac{x+b}{x}\right] \quad (4)$$

As  $x \rightarrow 0$ , this result diverges (slowly, like  $\text{Log}[x]$ ), and hence the electric field at the corner is infinite.

2. Now let's turn to the full problem of a conducting tube. For the moment, we will assume that the tube has a constant charge density. From above, we know that the electric field at the corner must diverge due to the two faces that the point touches. The electric field from the other two faces is finite and also points in this same direction. Therefore, the electric field at the corner diverges.

Next, let us turn to a conducting tube and relax the assumption that the charge density  $\sigma$  is constant. In a conducting tube, the density would increase near the corners because of the self-repulsion of the charges. This has the effect of making the field even larger than the above reasoning would imply, so the above conclusion of a divergent field is still valid. In other words, this conclusion is true for both a conducting and nonconducting tube.

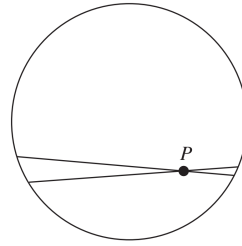
More generally, this type of divergence happens at any corner. For example, if you place a charge  $Q$  on a conducting cube, the electric field will diverge at the corners. As you can show, the divergence found above for the infinite strip does not occur as  $x \rightarrow 0$  because the rods were infinitely long, but rather because of the bits of charge right on the edge of the strip, which will come infinitely close to our point of interest as  $x \rightarrow 0$ . This divergence occurs even when the angle at the corner is not  $90^\circ$  - it would also be true for a polygon with 100 sides, in which the surface bends by only a few degrees at each corner. This divergence also happens at a kink in a wire.  $\square$

## Conducting Disk

### Example

There is a very sneaky way of finding the charge distribution on a conducting circular disk with radius  $R$  and charge  $Q$ . Our goal is to find a charge distribution such that the electric field at any point in the disk has zero component parallel to the disk.

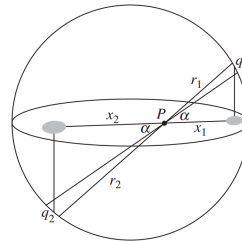
Recall from Problem 1.17 in the textbook (which you did in Homework 5) that the field at any point  $P$  inside a spherical shell with uniform surface charge density is zero, since the contributions due to both cones emerging from  $P$  cancel.



Consider the projection of this shell onto the equatorial plane containing  $P$ . Explain why this setup is relevant, and use it to find the desired charge density on a conducting disk.

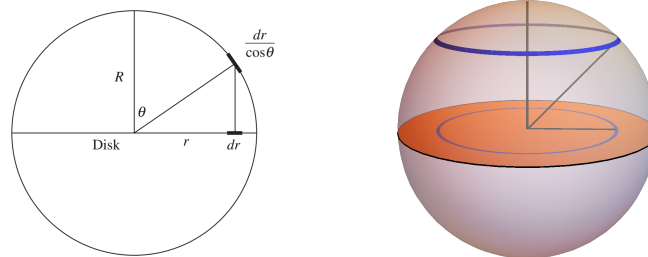
### Solution

Recall the argument in Problem 1.17 that showed why the field inside a spherical shell is zero. In the figure below, the two cones that define the charges  $q_1$  and  $q_2$  on the surface of the shell are similar, so the areas of the end patches are in the ratio of  $\left(\frac{r_1}{r_2}\right)^2$ . This factor exactly cancels the  $\frac{1}{r^2}$  effect in Coulomb's law, so the fields from the two patches are equal and opposite at point  $P$ . The field contributions from the entire shell therefore cancel in pairs.



Now project the charges residing on the upper and lower hemispheres onto the equatorial plane containing  $P$ . The charges  $q_1$  and  $q_2$  in the patches mentioned above end up in the shaded patches shown in the figure above. The (horizontal) fields at point  $P$  from these shaded patches have magnitudes  $\frac{kq_1}{x_1^2}$  and  $\frac{kq_2}{x_2^2}$ . But due to the similar triangles in the figure,  $x_1$  and  $x_2$  are in the same ratio as  $r_1$  and  $r_2$ . Hence the two forces have equal magnitudes, just as they did in the case of the spherical shell. The forces from all of the various parts of the disk therefore again cancel in pairs, so the horizontal field is zero at  $P$ . Since  $P$  was arbitrary, we see that the horizontal field is zero everywhere in the disk formed by the projection of the charge from the original shell. We have therefore accomplished our goal of finding a charge distribution that produces zero electric field component parallel to the disk.

Since the spherical shell has a larger slope near the sides, more charge is above a given point in the equatorial plane near the edge, compared with at the center. The density of the conducting disk therefore grows with  $r$ , as shown in the figure below.



To be quantitative, we want to find the charge density  $\sigma_{\text{disk}}[r]$  when we collapse the sphere with uniform charge density  $\sigma_{\text{sphere}}$  onto the disk. The charge on an infinitesimal patch of the disk at a given point  $(r, \phi)$  will be

$\sigma_{\text{disk}}[r] (r dr d\phi)$ . Using the diagram above, the patch on the sphere above the point  $(r, \phi)$  will have length  $\frac{dr}{\cos[\theta]}$ , width  $r d\phi$ , and therefore a total charge of  $2 \sigma_{\text{sphere}}(r d\phi) \left(\frac{dr}{\cos[\theta]}\right)$ . Note that the factor of 2 emerges since we collapse both this patch and its symmetric counterpart at  $\theta \rightarrow \pi - \theta$ . Equating these two charges, we find

$$\sigma_{\text{disk}}[r] = \frac{2 \sigma_{\text{sphere}}}{\cos[\theta]} \quad (5)$$

To remove the  $\cos[\theta]$ , we note that  $r = R \sin[\theta]$  so that  $\cos[\theta] = \left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2}$ , and hence

$$\sigma_{\text{disk}}[r] = \frac{2 \sigma_{\text{sphere}}}{\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2}} \quad (6)$$

As a double check, we can use  $\sigma_{\text{sphere}} = \frac{Q}{4\pi R^2}$  and ensure that the total charge on the disk will be  $Q$ ,

$$\begin{aligned} \int_0^{2\pi} \int_0^R \frac{2 \sigma_{\text{sphere}}}{\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2}} r dr d\phi &= 4\pi \sigma_{\text{sphere}} \int_0^R \frac{r}{\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2}} dr \\ &= 4\pi \sigma_{\text{sphere}} \left\{ -R^2 \left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2} \right\}_{r=0}^{r=R} \\ &= 4\pi \sigma_{\text{sphere}} R^2 \\ &= Q \end{aligned} \quad (7)$$

The density diverges at the edge of the conducting disk, but the total charge has the finite value  $Q$ . It is fairly intuitive that the density should grow as  $r$  increases, because charges repel each other toward the edge of the disk. However, one should be careful with this type of reasoning. In the lower-dimensional analog involving a one-dimensional rod of charge, the density is actually essentially uniform, all the way out to the end; see the next problem.  $\square$

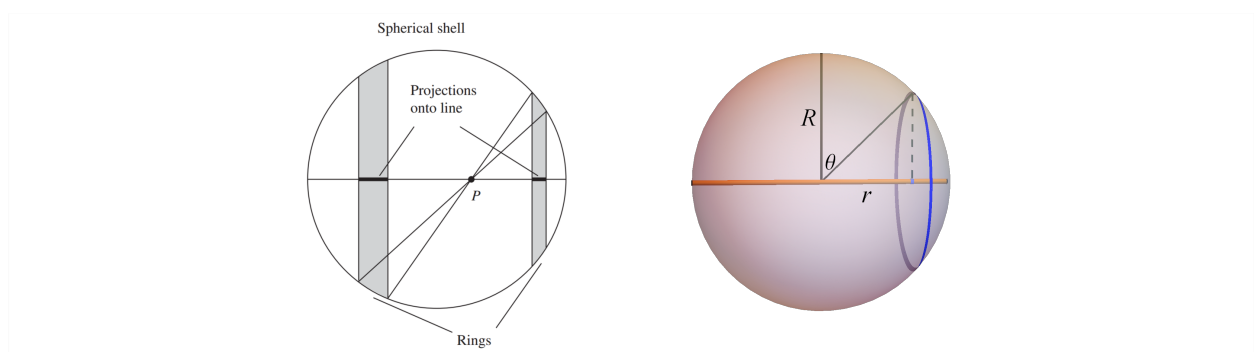
## Conducting Stick

### Example

The same strategy from the previous problem can be used to find the charge distribution on a conducting stick. Consider a stick of length  $2R$  centered at the origin and lying on the  $x$ -axis. To find the charge  $\lambda[x] dx$  which lies between  $x$  and  $x + dx$  on the stick, consider a sphere of radius  $R$  centered at the origin. Collapse all of the charge between  $x$  and  $x + dx$  onto the  $x$ -axis, and find the charge distribution  $\lambda[x]$  of the conducting stick. Then explain why this answer is absolutely absurd (and yet still correct!).

### Solution

We follow the same strategy as for the [Conducting Disk](#), namely, to take a sphere of radius  $R$  and collapse it down onto a rod. We know that each patch on the two cones emerging from  $P$  will cancel each other, and this cancellation continues after we project the two patches onto the  $x$ -axis (see below, left). Therefore, the charge  $\lambda[x] dx$  between  $x$  and  $x + dx$  on the rod will be given by the total charge on the ring of the charged sphere between  $x$  and  $x + dx$  (below, right).



As we saw above for the [Conducting Disk](#), the diagonal length of a ring at a distance  $x$  from the center of the sphere is  $\frac{dx}{\cos[\theta]}$ , and the circumference of this ring (which has radius  $(R^2 - x^2)^{1/2}$ ) is  $2\pi(R^2 - x^2)^{1/2}$ . Since  $x = R \sin[\theta]$ ,  $\cos[\theta] = (1 - (\frac{x}{R})^2)^{1/2}$  so that the total charge on this ring is given by

$$\lambda[x] dx = 2\pi \sigma_{\text{sphere}} (R^2 - x^2)^{1/2} \frac{dx}{(1 - (\frac{x}{R})^2)^{1/2}} = 2\pi R \sigma_{\text{sphere}} dx \quad (8)$$

As a quick double check, the full charge over the entire sphere assuming  $\sigma_{\text{sphere}} = \frac{Q}{4\pi R^2}$  is

$$\int_{-R}^R \lambda[x] dx = 4\pi R^2 \sigma_{\text{sphere}} = Q \quad (9)$$

as expected. Equation (8) implies that the charge density on a conducting rod is

$$\lambda[x] = 2\pi R \sigma_{\text{sphere}} = \frac{Q}{2R} \quad (10)$$

Shockingly, this is *uniform*! This is an absolutely astonishing result!!!

In case your jaw isn't hanging open, consider how unintuitive this result is. A point at the center of a uniformly charged rod will certainly feel zero force by symmetry, but *any point not in the exact center* will feel a force pointing away from the center there will be more charged rod on one of its sides than the other. As a particularly troublesome case, consider the force on the very ends of the rod - clearly these must point outwards. So how can this charge density be correct?

As a fun historical aside, there was a very lively debate over the charge density of a conducting stick. Part of the problem was the apparently ridiculous nature of the uniform charge density solution. Another issue is that the charge density on a conducting needle seemed to be different depending on which model you use (see [Charge on a Conducting Needle](#) by Griffiths and Li 1996 for an excellent discussion of this problem).

This problem is discussed extensively in Problem 3.5 of the textbook. Loosely speaking, it turns out that if you correct the uniform charge density by breaking up the stick into  $N$  small segments and letting all of the segments (except the two end segments) move to their equilibrium positions, then in the limit  $N \rightarrow \infty$  the movement required by each segment will shrink to zero. In this limit, the two end segments can be ignored since they are single points on a 1D rod. You are invited to think about this perplexing problem and read over the fantastic description in the text.  $\square$

## Image Charge of a Conducting Cylinder

### Maximum Field from a Blob

## Symmetry and Superposition

# Mathematica Initialization